A RICCI NILSOLITON IS NONGRADIENT

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In this brief note, we clarify that a Ricci nilsoliton cannot be of gradient type. The key is the rigidity theorem for gradient homogeneous Ricci solitons proved by Petersen and Wylie in [1]. Recall that a gradient Ricci soliton is a Riemannian metric g togheter with a function f such that

$$\mathrm{Ric}_g + \mathrm{Hess} f = \lambda g, \tag{1}$$

where λ is the soliton constant.

In [2], Lauret proves the existence of many left invariant Ricci solitons on nilpotent Lie groups. The first explicit construction of Lauret solitons has been obtained by Baird and Danielo in [4]. In particular they show that the soliton structure on Nil^3 is of nongradient type. Remarkably this is the first example of nongradient Ricci soliton. In [5], it is proved that any left invariant gradient Ricci soliton must have a nontrivial Euclidean de Rham factor. As an application of this result it is shown that any generalized Heisenberg Lie group is a nongradient left invariant Ricci soliton. In [1], Petersen and Wylie study the rigidity of gradient Ricci solitons with symmetry. Recall that a simply connected gradient Ricci soliton is called rigid if it is isometric to $N \times \mathbb{R}^k$ where N is an Einstein manifold and $f = \frac{\lambda}{2} |x|^2$ on the Euclidean factor. Among many other results, Petersen and Wylie prove the following optimal result for homogeneous gradient Ricci solitons.

Theorem 0.1. *(Petersen-Wylie) All homogeneous gradient Ricci solitons are rigid.*

Using this nice result we can now prove the following optimal generalization of theorem 3.7 in [5].

Theorem 0.2. *A nilsoliton cannot be of gradient type.*

Proof. Let $\{\mathfrak{n}, \langle,\rangle\}$ be a nilpotent soliton metric Lie algebra. Now, if the soliton is assumed to be of gradient type it must be isometric to a Riemannian product $N \times \mathbb{R}^k$ where N is an Einstein manifold. This implies that the Ricci tensor of our soliton is semi-definite. This is absurd with that fact any left invariant metric over a noncommutative nilpotent Lie group must have mixed Ricci curvatures, see theorem 2.4 in [3]. □

1. Final remarks

Theorem 0.2 together with the results in [5] provide a fairly complete description of the geometry of a Ricci nilsoliton. It would be now interesting to classify all the nilpotent Lie algebras that admit a soliton structure. In particular it would be interesting to understand if the characteristically nilpotent condition is the only obstruction to the existence of a soliton metric. For more details we refer to [2], [5].

References

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